Journal of Sound and Vibration (1999) **225**(5), 845–856
Article No. jsvi.1999.2274, available online at http://www.idealibrary.com on IDEAL®

COMMENTS ON NON-LINEAR FORMULATIONS FOR TRAVELLING STRING AND BEAM PROBLEMS

H. KOIVUROVA

Department of Mechanical Engineering, University of Oulu, Oulu, Finland

AND

E.-M. SALONEN

Laboratory of Theoretical and Applied Mechanics, Helsinki University of Technology, Helsinki, Finland

(*Received* 26 *November* 1998, *and in final form* 1 *March* 1999)

Kinematic aspects of non-linear travelling structure problems are reviewed and clarified in this paper. The description of the motion of a structure can be based on two different formulations; Eulerian or mixed Eulerian–Lagrangian. The mixed formulation leads to simpler equations but it requires comprehensive restrictions. These restrictions and the formulation itself are considered and explained. The inertia terms of the equations of motion are derived and are shown to be the same as those in the earlier papers.

1. INTRODUCTION

Axially moving material problems consider the dynamic response, vibration and stability of structural members that are in a state of translation. The problem is an extensive one that encompasses such apparently diverse mechanical systems as the paper web in a paper mill or printing press, on/off winding of textile fibres and paper, filament winding of composite components, high-speed magnetic tapes, power transmission chains and belts, band saws, pipes conveying fluid, etc. Recent developments in research on axially moving materials have been reviewed by Wickert and Mote [1], Arbate [2] and Païdoussis and Li [3].

The earliest non-linear theoretical studies were those of Zaiser [4], who analyzed the travelling string problem using a Eulerian description. He began from an application of Newton's second law to a control volume and obtained four simulataneous differential equations for the axial momentum, transverse momentum, mass tension relation and continuity. Similar formulations were later considered by Ames *et al.* [5] and Kim and Tabarrok [6].

Mote [7] developed an alternative formulation that included the elastic effect and geometric non-linearity by means of Hamilton's principle.* The transport

*In Hamilton's principle, the positions of a given set of particles is considered at two different times. In axially moving strings and beams different particles occupy the domain of interest [8].

velocity was taken as a constant. In a further derivation, he assumed the derivatives of longitudinal displacement to be small and ignored them. This led to a second-order equation, the transverse equation of motion. Later Thurman and Mote [9] and Wickert [10] used a similar formulation without the assumption for derivatives of longitudinal displacement, resulting in a governing equation for longitudinal and transverse motion.

The most notable difference between these two developments is the definition of displacement. Zaiser [4] and Ames *et al.* [5] used a pure Eulerian frame, in which they defined the displacement u of a particle of a string with respect to a spatial point, x, while Mote [7]. Thurman and Mote [9] and Wickert [10] used a mixed description. These traditional models represent the translating element as a taut string or beam that is drawn perfectly straight under tension. A similar mixed formulation has also been used for a travelling elastic cable (a loose string) by Perkins and Mote [11] and for a buckled beam by Hwang and Perkins [12].

The purpose of this paper is to review kinematics of the non-linear travelling structure problems. Since, in our opinion, none of the above papers present a complete definition or clear explanation for the mixed formulation used, the main objective is to define the formulation and explain some questions arising from the derivation of the equations and assumptions needed on account of this formulation. The description of displacement can be interpreted by defining a mixed Eulerian–Lagrangian formulation which is similar to the one presented by Vu-Quoc and Li [13] and Behdinan *et al.* [14] for a sliding beam problem. The definition of the formulation is straightforward for the taut string and straight beam problems, but the association of the spatial point and the material particle makes the situation more complicated in the case of initially curved structures.

2. AN EULERIAN–LAGRANGIAN FORMULATION FOR THE TAUT STRING AND STRAIGHT BEAM PROBLEMS

In the traditional Eulerian formulation the variable $u(x, t)$ describes the transverse displacement of the string material element instantaneously located at x, as shown in Figure 1(a). The description used in references [7, 9, 10] differs from the traditional Eulerian description in possessing a longitudinal component u_x [see Figure 1(b)]. Mote [7] explains that u_x is the longitudinal displacement and that it is related to coordinates translating at speed c, while the transverse component u_y is related to a spatial frame. Wickert [10] suggested later that both components, the transverse as well as the longitudinal are described in the spatial frame. Moreover, he added that longitudinal and transverse displacement must be coupled for finite amplitude motion. Thurman and Mote [9] made no comment on the description of longitudinal displacement or its definition.

The description used above can be explained using a similar formulation to that in references [13, 14] for a sliding beam problem. These two references present a dynamic formulation for sliding beams that are deployed or retrieved through prismatic joints, distinguishing different configurations: the initial un-deformed or material configuration, a spatially fixed intermediate configuration and a sliding deformed configuration. The intermediate configuration is an *artifice* introduced

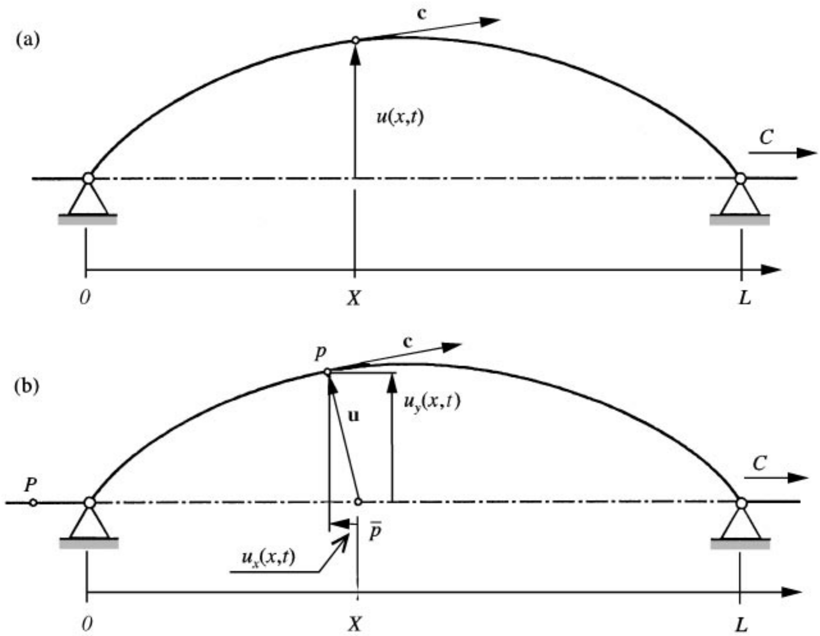

Figure 1. Schematic models for axially moving string formulations (a) Eulerian description [5], (b) "mixed" description [9].

for the purposes of the formulation, and it can be seen as Eulerian domain with respect to the translating un-deformed beam and a Lagrangian domain with respect to the current deformed beam.

For the problems of taut axially moving strings and straight beams, the intermediate configuration can be chosen as straight under steady state conditions. The choice is quite obvious, since the flow of the material particles is stationary and the geometry is known, the association between a spatial point and a material particle is specified. The intermediate state can be seen as an Eulerian domain with respect to the translating structure, since the material point P coincides with the fixed point $\bar{p}$ at time t, and as a Lagrangian domain with respect to the current deformed structure, since the deformation is described from the fixed point $\bar{p}$ to the spatial point p, see Figure 1(b).

When a stretched steady state is used as an intermediate configuration, some restrictions must be made to ensure proper one-to-one mapping of the material particle to the intermediate configuration and the deformed final configuration. First of all, the transport motion of the material particles should be such that the structure has a prescribed steady state. This is so if the axial speed c is a constant in time and the flow is stationary in this steady position, that is, at a given point with intermediate configuration, velocity does not change with time. Moreover, the longitudinal component u_x of the displacement can arise only through the transverse motion, as mentioned in reference [9].

The benefits of a mixed Eulerian–Lagrangian description relative to the pure Eulerian description are quite obvious. The large deflection from the intermediate configuration to a spatial point is identical to the displacement of a material particle in the string or beam in the absence of axial motion. Thus, the strain energy needed in Hamilton's principle is identical in form to the strain energy of a non-moving non-linear structure. Accordingly, the parts of the equations of motion which do not include time derivatives are also the same. The equation of motion can be derived using an equation for the motion of a non-moving structure and replacing the time derivatives with the material time derivative in the mixed formulation. The equations of motion for a large amplitude vibrating non-moving string [18] is

$$EAu_{y,xx} - (EA - T_0)\frac{(1 + u_{x,x})^2 u_{y,xx} - u_{y,x}(1 + u_{x,x})u_{x,xx}}{[(1 + u_{x,x})^2 + u_{y,x}^2]^{3/2}} = \rho\,\frac{\mathrm{d}^2 u_y}{\mathrm{d}t^2},$$

$$EAu_{x,xx} - (EA - T_0)\frac{u_{y,x}^2\, u_{x,xx} - u_{y,x}(1 + u_{x,x})u_{y,xx}}{[(1 + u_{x,x})^2 + u_{y,x}^2]^{3/2}} = \rho\,\frac{\mathrm{d}^2 u_x}{\mathrm{d}t^2}, \tag{1}$$

where various symbols have their conventional meanings, (E is Young's modulus, A the area of cross-section, T_0 the initial tension and ρ the density of the string). The problem is thus to derive detailed expressions for the right-hand-side terms. It should noted, however, that the equations (1) represent the motion in terms of Lagrangian variables. There is a fixed set of particles associated with the equation, and therefore the state of the material is known and the parameters E, A and ρ can be described with respect to the co-ordinate x. On the other hand, equations (9) to derived later represents motion in terms of Eulerian variables, which are only defined within a specified spatial domain for a set of material particles which enter and leave the domain. The equation is valid at an instant but can still be re-applied at every instant [16].

The velocity vector $\mathbf{v}$ of a material particle and its material time derivative is next considered, (that is, its acceleration vector $\mathbf{a}$) using the differential geometry shown in Figure 2. The motion of a material particle P is observed during an infinitesimal time step $\mathrm{d}t$. It possess the spatial point p at time t and point q at time $t + \mathrm{d}t$. Since the time step is infinitesimal, the displacement can be written as $\mathbf{v}\,\mathrm{d}t$ where $\mathbf{v}$ is the velocity at point p. If we map the fixed points in the Eulerian domain (steady state) corresponding to points p and q, they are $\bar{p}$ and $\bar{q}$. Under steady state conditions, a material particle would move the distance $\mathrm{d}x$ in the time $\mathrm{d}t$ with a constant speed c, hence

$$\mathrm{d}x = c\,\mathrm{d}t. \tag{2}$$

Since the displacement of a material particle from the steady configuration to the deformed configuration [that is, from point $\bar{p}$ (located at co-ordinate x) to the spatial point p] is $\mathbf{u} = \mathbf{u}(\bar{p}, t) = \mathbf{u}(x, t)$, which means that x and t are used as independent variables. The displacement of a material particle from point $\bar{q}$ is therefore

$$\begin{aligned}\mathbf{u}(x + \mathrm{d}x, t + \mathrm{d}t) &= \mathbf{u}(x, t) + \mathrm{d}\mathbf{u} = \mathbf{u}(x, t) + \mathbf{u}_{,x}(x, t)\mathrm{d}x + \mathbf{u}_{,t}(x, t)\mathrm{d}t \\ &= \mathbf{u}(x, t) + \mathbf{u}_{,x}(x, t)c\mathrm{d}t + \mathbf{u}_{,t}(x, t)\mathrm{d}t.\end{aligned} \tag{3}$$

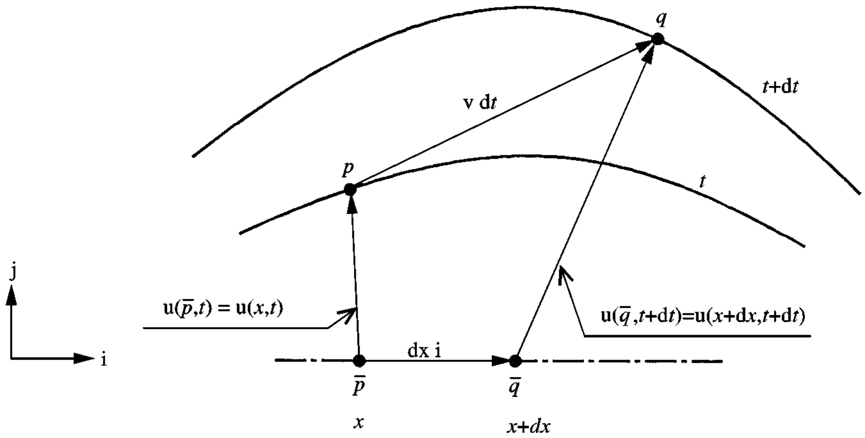

Figure 2. Displacement of a material particle at the two points in time.

This is obtained by means of the chain rule and equation (2). Finally, if the geometry of Figure 2 uses a vector equation

$$\mathbf{u}(x, t) + \mathbf{v}\mathrm{d}t = \mathrm{d}x\mathbf{i} + \mathbf{u}(x + \mathrm{d}x, t + \mathrm{d}t) \tag{4}$$

can be developed. Substituting equations (2) and (3) into equation (4), the expected result is obtained:

$$\mathbf{v} = \mathbf{u}_{,t} + c(\mathbf{i} + \mathbf{u}_{,x}). \tag{5}$$

The velocity of a material particle is composed of two parts: the transport motion of the string $\mathbf{c} = c(\mathbf{i} + \mathbf{u}_{,x})$ and the motion of the configuration $\mathbf{u}_{,t}$.

The acceleration of a material particle in the string can be developed from the change in velocity using the chain rule [equations (2) and (5)], that is

$$\begin{aligned} \mathrm{d}\mathbf{v} &= \mathbf{v}_{,x}\mathrm{d}x + \mathbf{v}_{,t}\mathrm{d}t = [\mathbf{u}_{,t} + c(\mathbf{i} + \mathbf{u}_{,x})]_{,x}\, c\mathrm{d}t + [\mathbf{u}_{,t} + c(\mathbf{i} + \mathbf{u}_{,x})]_{,t}\, \mathrm{d}t \\ &= (\mathbf{u}_{,tt} + 2\, c\, \mathbf{u}_{,xt} + c^2\mathbf{u}_{,xx})\mathrm{d}t. \end{aligned} \tag{6}$$

Thus, dividing the change in velocity by dt, the acceleration vector

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{v}_{,t} + c\mathbf{v}_{,x} = \frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} = \mathbf{u}_{,tt} + 2c\mathbf{u}_{,xt} + c^2\mathbf{u}_{,xx} \tag{7}$$

is obtained.

The operator of the material time derivative of the mixed Eulerian–Lagrangian description for an axially travelling taut string and straight beam is thus

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}, \tag{8}$$

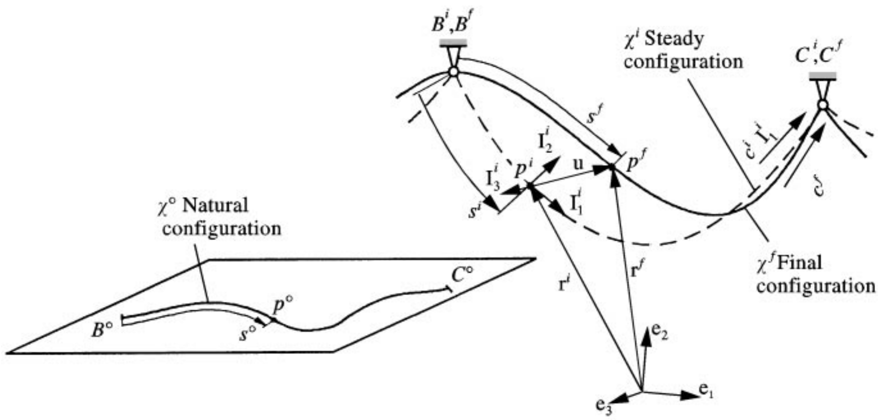

Figure 3. Diagram defining the natural, steady and final configurations of a cable and the displacement in a point in the configuration.

and by substituting equation (7) into equation (1) the equations of motion for an axially moving string can be obtained:

$$EAu_{y,xx} - (EA - T_0)\frac{(1 + u_{x,x})^2 u_{y,xx} - u_{y,x}(1 + u_{x,x})u_{x,xx}}{[(1 + u_{x,x})^2 + u_{y,x}^2]^{3/2}}$$
$$= \rho(u_{y,tt} + 2cu_{y,xt} + c^2 u_{y,xx}),$$
$$EAu_{x,xx} - (EA - T_0)\frac{u_{y,x}^2 u_{x,xx} - u_{y,x}(1 + u_{x,x})u_{y,xx}}{[(1 + u_{x,x})^2 + u_{y,x}^2]^{3/2}}$$
$$= \rho(u_{x,tt} + 2cu_{x,xt} + c^2 u_{x,xx}). \tag{9}$$

These are the same as given in reference [8] for a beam, if the flexural stiffness terms are excluded.

3. THE LOOSE STRING PROBLEM (CURVED STEADY STATE)

An even more advanced formulation of this kind is presented by Perkins and Mote [11], who considered the vibration of a travelling elastic cable (a loose string). They defined displacement as a vector from the steady configuration† to a final configuration (see Figure 3), which is said to "represent the three-dimensional motion of the final configuration, and it is distinguished from the motion of a cable particle which includes the particle transport velocity", meaning that the velocity of a particle at the point p^f, denoted by $\mathbf{v}^f$, consists of the movement of the configuration $\mathbf{u}$ and the transport motion $\mathbf{c}^f$,

$$\mathbf{v}^f = \mathbf{u}_{,t} + \mathbf{c}^f. \tag{10}$$

In their case, the steady configuration is not straight but its shape depends on external loads and the transport speed c^i, and it is fixed in space. The string in the

†The word equilibrium is used in reference [10] rather than steady state.

natural configuration χ^o is un-stressed, and therefore it can be in transport motion only if the motion does not cause any stress.

The description of displacement can be explained with a mixed Eulerian–Lagrangian formulation similar to that given above for taut string and straight beam problems, although the associations between material particles and configuration points are not very clear. If the steady state itself and the flow of the material particles in it are known, it is possible to choose this as an intermediate configuration. The steady state configuration can be calculated form the natural state, as shown in reference [11], by using equations of equilibrium

$$-(\rho^i A^i (c^i)^2)_{,s} + T^i_{,s} = \rho^i A^i g l_1, \quad -\rho^i A^i K^i (c^i)^2 + K^i T^i = \rho^i A^i g l_2, \tag{11}$$

a constitutive equation

$$T^i = EA^0 \frac{1}{2}\left(\left(\frac{\mathrm{d}s^i}{\mathrm{d}s^0}\right)^2 - 1\right), \tag{12}$$

and the conservation of mass

$$\rho^0 A^0 \mathrm{d}s^0 = \rho^i A^i \mathrm{d}s^i = \rho^f A^f \mathrm{d}s^f, \quad \rho^0 A^0 c^0 = \rho^i A^i c^i = \rho^f A^f c^f, \tag{13}$$

where $(\ )_{,s}$ denotes differentiation with respect to the arc length s^i, ρ is the density, A is the cross sectional area, E is Young's modulus, T is the cable tension, K is the curvature of a configuration, g is the acceleration due to gravity and l_1 and l_2 are the components of a vertical base vector $\mathbf{e}_2 = l_1\mathbf{l}^i_1 + l_2\mathbf{l}^i_2$, where $\mathbf{l}^i_1$ and $\mathbf{l}^i_2$ are unit vectors directed along the cable and perpendicular to it. The equations of equilibrium can be derived by reference to the conservation of linear momentum (see Appendix A), without using the results of this formulation itself, as is the case in reference [11]. The flow of material particles in a steady state is also known by means of the equations of equilibrium, constitutive equation and conservation of mass, and according to reference [11] it is

$$c^i = c^o\left(\frac{2T^i}{EA^o} + 1\right)^{1/2}, \tag{14}$$

where c^o is the speed of transport in a situation where the transport motion does not cause any stress and therefore no deformation. Such a state can be reached when a string is placed on a "frictionless table", for instance, and it is exactly straight and undeformed.

The association between a material particle and a spatial point p^i in the steady configuration is now clear, but to ensure that an association exists with the point p^f in the final configuration, an assumption has to be made. Actually, this had already been done earlier when the expressions of conservation of mass were written. The first of equations (13) requires that the infinite pieces $\mathrm{d}s^f$ and $\mathrm{d}s^i$ have the same mass and therefore the same number of material particles as the piece $\mathrm{d}s^o$ of the natural configuration χ^o. In other words, the pieces $\mathrm{d}s^f$ and $\mathrm{d}s^i$ are "clones" of the base piece $\mathrm{d}s^o$, implying that if they are cut off from the cable and then placed on a "frictionless table" next to the base piece $\mathrm{d}s^o$, after removal of all stresses (and therefore all deformations) the pieces will look alike (Figure 4). This can be generalized to the

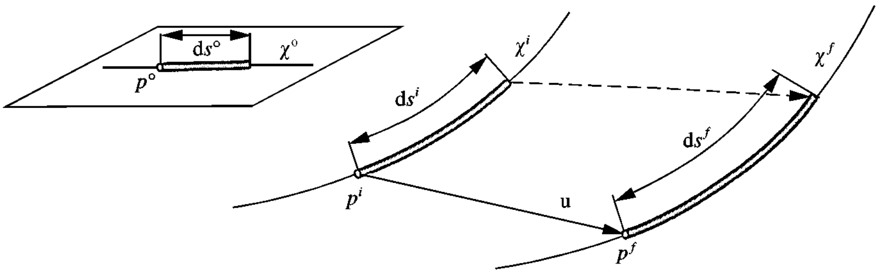

Figure 4. Deformation of a infinitesimal piece of a cable configuration.

entire cable by stating that the arc length s^i is a sufficient measure to identify a particle of the cable if the flow of particles is homogeneous.

Accepting the above assumption, the curved steady state can be chosen as the intermediate configuration and the equations of motion expressed with respect to its co-ordinates. The mixed formulation can be described in the same way as in the previous section, the intermediate state being regarded as a Eulerian domain with respect to the translating structure, since the material particle coincides with a fixed point p^i at time t, and as a Lagrangian domain with respect to the current deformed structure, since the deformation is described from the fixed point p^i to the spatial point p^f (see Figure 3). The advantage of this mixed Eulerian–Lagrangian formulation over Eulerian description is the same as in the straight string problem: the description simplifies the strain energy of a travelling cable to the form of a non-moving structure.

Another detail to be considered is the derivation of the expression of velocity $\mathbf{v}^f$ for a material particle (10) and the transport velocity

$$\mathbf{c}^f = c^f \mathbf{l}_1^f = c^i[\mathbf{l}_1^i + \mathbf{u}(s^i, t)_{,s}] \tag{15}$$

in the final configuration. Perkins and Mote [11] used the equation for the local tangential base vector in the final configuration

$$\mathbf{l}_1^f = \frac{\partial \mathbf{r}^f(s^i, t)}{\partial s^f} = [\mathbf{l}_1^i + \mathbf{u}(s^i, t)_{,s}]\frac{\mathrm{d}s^i}{\mathrm{d}s^f} \tag{16}$$

and two equations of mass conservation

$$A^i \mathrm{d}s^i = A^f \mathrm{d}s^f \tag{17}$$

and

$$A^i c^i = A^f c^f \tag{18}$$

to form equation (15), but they did not take into account the change in density in this equation. The first equation of mass conservation is obvious if density is considered to be constant, but it is unclear how the second equation was derived. Actually, it is possible to obtain equation (15) using the kinematics of the cable alone.

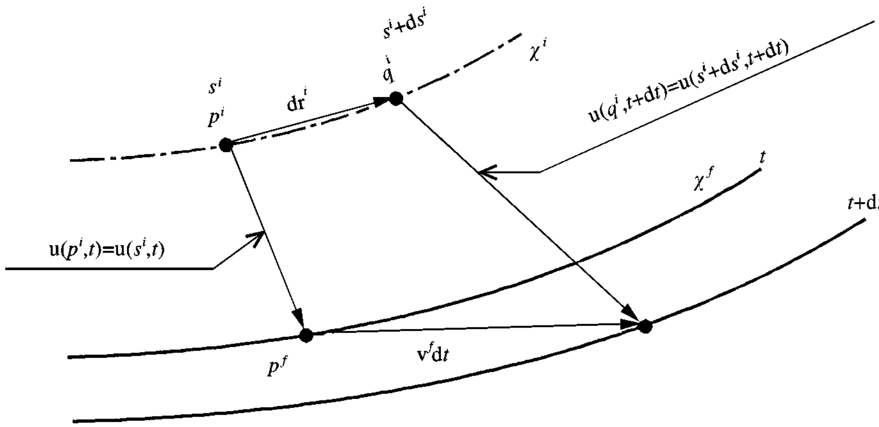

Figure 5. Displacement of a point in the configuration, and movement of a material particle.

The motion of a particle is traced during an infinitesimal time step dt from the point p^i in the steady state and from the corresponding sample point p^f in the final state, as shown in Figure 5. The displacement of a particle in the steady state consists of transport motion only, and is therefore

$$\mathrm{d}\mathbf{r}^i = c^i \mathbf{l}_1^i \mathrm{d}t. \tag{19}$$

The displacement of a particle from the final state consists of the transport motion and the motion of the configuration, according to equation (10), and is now $\mathbf{v}^f \mathrm{d}t$. Since the displacement of the sample point p^i, the point in the configuration where the particle is located at time t, is $\mathbf{u} = \mathbf{u}(s^i, t)$, it is clear that s^i and t are used as independent variables. The displacement of a configuration point p^i where the particle under examination is located at time $t + \mathrm{d}t$ is therefore

$$\begin{aligned}\mathbf{u}(s^i + \mathrm{d}s^i, t + \mathrm{d}t) &= \mathbf{u}(s^i, t) + \mathrm{d}\mathbf{u} = \mathbf{u}(s^i, t) + \mathbf{u}_{,s}(s^i, t)\mathrm{d}s + \mathbf{u}_{,t}(s^i, t)\,\mathrm{d}t \\ &= \mathbf{u}(s^i, t) + \mathbf{u}_{,s}(s^i, t)c^i\mathrm{d}t + \mathbf{u}_{,t}(s^i, t)\,\mathrm{d}t. \end{aligned}\tag{20}$$

The result was obtained by means of the chain rule and a scalar version of equation (19), or $\mathrm{d}s^i = c^i\mathrm{d}t$. Finally, if the geometry of Figure 5 is used, a vector equation

$$\mathbf{u}(s^i, t) + \mathbf{v}^f\mathrm{d}t = \mathrm{d}\mathbf{r}^i + \mathbf{u}(s^i + \mathrm{d}s^i, t + \mathrm{d}t) \tag{21}$$

can be developed.
Substituting equations (10), (19) and (20) into equation (21), the expected result is obtained:

$$\mathbf{c}^f = c^i(\mathbf{l}_1^i + \mathbf{u}_{,s}). \tag{22}$$

Applying equation (16), the transformation equation can be obtained in the form

$$\mathbf{c}^f = c^i \frac{\mathrm{d}s^f}{\mathrm{d}s^i} \mathbf{l}_1^f = c^f\, \mathbf{l}_1^f, \tag{23}$$

which shows that the transport velocity is indeed in a direction tangential to the final configuration, which, taken together with the first equation of mass conservation (17), results in the second equation of mass conservation (18). Thus, equations (15) and (23) are consequences of kinematics only, and the second equation of mass conservation (18) follows from them and not *vice versa*.

The acceleration of a particle in the cable can be derived from the change in velocity using the chain rule, a scalar version of equations (19) and (10); That is

$$\begin{aligned} \mathrm{d}\mathbf{v}^f &= \mathbf{v}^f_{,s}\,\mathrm{d}s^i + \mathbf{v}^f_{,t}\,\mathrm{d}t = \mathbf{v}^f_{,s}\,c^i\,\mathrm{d}t + \mathbf{v}^f_{,t}\,\mathrm{d}t \\ &= (\mathbf{u}_{,ts} + \mathbf{c}^f_{,s})c^i\,\mathrm{d}t + (\mathbf{u}_{,tt} + \mathbf{c}^f_{,t})\,\mathrm{d}t. \end{aligned} \tag{24}$$

Thus, if the change in velocity is divided by dt and equation (15) is substituted, we obtain the acceleration vector

$$\begin{aligned} \mathbf{a}^f &= \frac{\mathrm{d}\mathbf{v}^f}{\mathrm{d}t} = \mathbf{v}^f_{,t} + c^i\,\mathbf{v}^f_{,t} = \frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} = \mathbf{u}_{,tt} + c^i\,\mathbf{u}_{,st} + c^i\,\mathbf{c}^f_{,s} \\ &= \mathbf{u}_{,tt} + c^i\mathbf{u}_{,st} + [c^i(\mathbf{l}^i_1 + \mathbf{u}_{,s})]_{,t} + c^i[c^i(\mathbf{l}^i_1 + \mathbf{u}_{,s})]_{,s} \\ &= \mathbf{u}_{,tt} + 2c^i\mathbf{u}_{,st} + c^i[c^i(\mathbf{l}^i_1 + \mathbf{u}_{,s})]_{,s}. \end{aligned} \tag{25}$$

By reducing this equation and substituting the component form of the displacement $\mathbf{u} = u_1\mathbf{l}^i_1 + u_2\mathbf{l}^i_2 + u_3\mathbf{l}^i_3$ into it, the same inertia terms as in the equations of motion in reference [10] are obtained.

4. CLOSING REMARKS

A review of the formulations of the axially travelling structure problems is presented. The main purpose is to clarify the earlier formulations of the mixed Eulerian–Lagrangian description, which did not define or explain the subject. The restrictions needed as a consequence of the formulation are also indicated. The flow of material particles has to be homogenous in order that the number of particles and their distribution remain the same. This requirement is due to the mapping of the material particles from the undeformed state through the intermediate configuration to the current deformed structure. Moreover, the inertia terms of the equations of motion are derived and it is shown that they are the same as in the earlier papers.

REFERENCES

1. J. A. Wickert and C. D. Mote, Jr 1988 *Journal of Shock and Vibration Digest* **20**, 3–13. Current research on the vibration and stability of axially-moving materials.
2. S. Arbate 1992 *Measurements and Machines Theory* **27**, 645–659, Vibrations of belts and belt drives.
3. M. P. Païdoussis and G. X. Li 1993 *Journal of Fluid and Structures* **7**, 137–204. Pipes conveying fluid: a model dynamical problem.
4. J. N. Zaiser 1964 *Ph.D. Dissertation, University of Delaware*. Nonlinear vibrations of a moving threadline.

5. W. F. AMES, S. Y. LEE and J. N. ZAISER 1968 *International Journal of Nonlinear Mechanics* **3**, 449–469. Nonlinear vibration of a travelling threadline.
6. Y. KIM and B. TABARROK 1972 *Journal of The Franklin Institute* **293**, 381–399. On the nonlinear vibration of travelling string.
7. C. D. MOTE, Jr 1996 *ASME Journal of Applied Mechanics* **33**, 463–464. On the nonlinear oscillation of an axially moving string.
8. B. TABARAK, C. M. LEECH and Y. I. KIM 1974 *Journal of The Franklin Institute* **297**, 201–220. On the dynamics of an axially moving beam.
9. A. L. THURMAN and C. D. MOTE Jr 1969 ASME *Journal of Applied Mechanics* **36**, 83–91. Free, periodic, nonlinear oscillation of an axially moving strip.
10. J. A. WICKERT 1992 *International Journal of Nonlinear Mechanics* **27**, 503–517. Nonlinear vibration of a travelling tensioned beam.
11. N. C. PERKINS and C. D. MOTE, Jr 1987 *Journal of Sound Vibration* **114**, 325–340. Three-dimensional vibration of travelling elastic cables.
12. S. J. HWANG and N. C. PERKINS 1992 *Journal of Sound and Vibration* **154**, 381–396. Supercritical stability of an axially moving beam, Part I: model and equilibrium analysis.
13. L. VU-QUOC and S. LI 1995 *Computer Methods in Applied Mechanics and Engineering* **120**, 65–118. Dynamics of sliding geometrically-exact beams. large angle maneuver and parametric resonance.
14. K. BEHDINAN, M. SYLIANOU and B. TABARROK 1997 *Journal of Sound and Vibration* **208**, 517–539. Dynamics of flexible sliding beams – non-linear analysis, Part I: formulation.
15. A. W. LEISSA and A. M. SAAD 1994 *ASME Journal of Applied Mechanics* **61**, 296–301. Large amplitude vibrations of strings.
16. A. A. RENSHAW, C. D. RAHN, J. A. WICKERT and C. D. MOTE, Jr 1998 *ASME Journal of Vibration and Acoustics* **120**, 634–636. Energy conserved functionals for axially moving materials.

APPENDIX A

The equation of equilibrium for the steady state of an axially moving loose cable can be derived using the conservation of linear momentum. Since the system has a fixed control volume and constant mass, the equation is

$$\mathbf{R} = \mathbf{q}_p, \tag{A1}$$

where $\mathbf{R}$ is the resultant of the external forces and $\mathbf{q}_p$ is the momentum flux through the boundaries.

The conservation of momentum for an infinitesimal piece of travelling cable $\mathrm{d}s^i$, shown in Figure 6, is

$$\begin{aligned} &P^i(s^i + \mathrm{d}s^i)\,\mathbf{l}_1^i\,(s^i + \mathrm{d}s^i) - P^i(s^i)\,\mathbf{l}_1^i\,(s^i) - \rho^i A^i\,\mathrm{d}s^i g\mathbf{e}_2 \\ &\quad = \rho^i A^i[c^i(s^i + \mathrm{d}s^i)]^2\mathbf{l}_1^i(s^i + \mathrm{d}s^i) - \rho^i A^i[c^i(s^i)]^2\mathbf{l}_1^i(s^i). \end{aligned} \tag{A2}$$

Dividing the equation by $\mathrm{d}s^i$ and using the definition of partial differentiation, gives

$$[P^i\,\mathbf{l}_1^i]_{,s} - \rho^i A^i g\mathbf{e}_2 = [\rho^i A^i (c^i)^2\,\mathbf{l}_1^i]_{,s} \tag{A3}$$

Moreover, if $\mathbf{e}_2 = l_1\mathbf{l}_1^i + l_2\mathbf{l}_2^i$, note that the derivative of the unit vector $\mathbf{l}_{1,s}^i = K^i\mathbf{l}_2^i$, where K^i is the curvature of χ^i, and separate the components in the directions of the

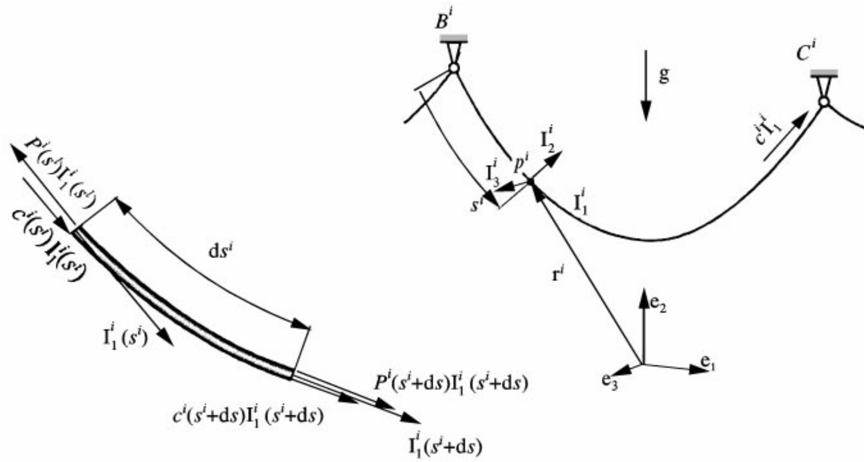

Figure 6. Diagram defining the steady configurations of a cable.

local co-ordinate system, the equation assumes the same form as presented by Perkins and Mote [11].

component $\mathbf{1}_1^i$:

$$-[\rho^i A^i (c^i)^2]_{,s} + P^i_{,s} = \rho^i A^i g l_1 \tag{A4}$$

component $\mathbf{1}_2^i$:

$$-\rho^i A^i K^i (c^i)^2 + K^i P^i = K^i P^i = \rho^i A^i g l_2. \tag{A5}$$